Numerical study of a three-dimensional generalized stadium billiard

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We study a generalized three-dimensional stadium billiard and present strong numerical evidence that this system is completely chaotic. In this convex billiard chaos is generated by the defocusing mechanism. The construction of this billiard uses cylindrical components as the focusing elements and thereby differs from the recent approach pioneered by Bunimovich and Rehacek [Commun. Math. Phys. **189**, 729 (1997)]. We investigate the stability of lower-dimensional invariant manifolds and discuss bouncing ball modes.

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Billiards are simple yet nontrivial models of classically chaotic systems. Of particular importance are Sinai's billiard [1] and Bunimovich's stadium [2] since they are proven to be completely chaotic and ergodic. These two billiards exhibit two different chaos generating mechanisms, namely, dispersion and defocusing. Upon scattering off a dispersing boundary element nearby trajectories diverge, and consecutive collisions with dispersing elements lead to an increasing divergence. In focusing billiards, nearby trajectories converge after a collision with the focusing boundary element. It is only after the trajectories pass through the focusing point that they start to diverge. Provided the free flight (including reflections at neutral boundary elements) is sufficiently long, the focusing may be overcompensated by the divergence and result in a defocusing. It is important to note that a weak focusing requires a long free flight before defocusing results. While dispersing billiards like Sinai's may easily be generalized to more than two dimensions, it was not until recently that completely chaotic higher-dimensional focusing billiards were constructed. Bunimovich and Rehacek proved that spherical caps attached to three-dimensional billiards with neutral boundary elements may be chaotic and ergodic [3]. The main difficulty to overcome was the weak focusing in directions transverse to the plane that is defined by consecutive scatterings of a trajectory with a spherical cap. Bunimovich and Rehacek solved this problem by putting certain conditions on the size and distance of spherical caps. This ensures that any focusing eventually turns into defocusing. However, numerical and analytical studies showed that these conditions may be relaxed [4], and that the mechanism of defocusing also works beyond three dimensions [5].

In this work we want to study a different construction of chaotic focusing billiards in three dimensions. The basic idea is as follows. Let us use cylindrical instead of spherical components as the focusing boundary elements. A cylindrical element focuses in two-dimensional planes perpendicular to the cylinder axis and is neutral in the directions of the axis. This avoids the problem caused by the weak focusing in spherical caps. The generation of high-dimensional chaos requires, however, more than one cylindrical boundary element, and their axes must be differently oriented. As an example we study numerically a three-dimensional generalization of the stadium billiard. Its construction is related to the construction of geodesic flows on high-dimensional (boundaryless) manifolds that are products of lower-dimensional manifolds [6]. Because of its use of cylindrical focusing elements it is also related to the billiard model of a self-bound three-body system [7]. Unlike the latter, the generalized three-dimensional stadium can easily be realized experimentally. This is of particular interest in view of recent experiments that investigate wave chaos in three-dimensional elastomechanical systems [8,9] and microwave cavities [10–12].

In addition to these possible applications the results presented in this work are also of interest to further fields in physics. We recall that billiards are being widely studied in the field of quantum chaos (for a review, see, e.g., Ref. [13]), and play an important role in extending this field to systems with more than two degrees of freedom [4,7,14,15]. They are also central to the investigation of three-dimensional wave chaos in resonant optical cavities [16].

Let us consider the three-dimensional billiard depicted in Fig. 1. We denote the radii of the lower and upper half cylinders as r_1 and r_2, respectively, and the distance between these half cylinders as $2a$. This billiard can be viewed as a generalization of the two-dimensional stadium to three dimensions. Its sections with planes normal to the x and y axes are partly desymmetrized two-dimensional stadia. One might expect this system to display high-dimensional chaos, since [17] ''In many-dimensional billiards with chaotic behavior the local instability has to be in all its two-dimensional sections.'' Note that the upper and lower half cylinders defocus

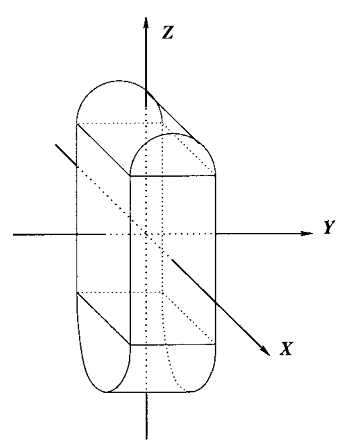

FIG. 1. Three-dimensional generalization of the stadium billiard.

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in the planes normal to the x axis and the y axis, respectively. Note further that this property persists even when the distance between the two half-cylinders approaches $2a=0$.

In what follows let us fix $r_1=r_2=r$ and $a=0$. To compute the Lyapunov spectrum we draw 10^4 uniformly distributed phase space points at random, fixing the velocity $|\vec{v}|=1$. We follow the time evolution for each of these phase space points for about 5×10^5 bounces off the boundary and compute the Lyapunov spectrum from the tangent map [18,19]. For the time evolution we note that the particle moves freely inside the billiard and undergoes specular reflections upon collisions with the boundary. Let $\vec{v}$ and $\vec{v}\,'$ denote the velocity immediately before and after a collision with the boundary, respectively. One has $\vec{v}\,'=\vec{v}-2(\vec{v}\cdot\vec{n})\vec{n}$, where $\vec{n}$ is the unit normal vector of the boundary at the collision point. The tangent map is a product of maps corresponding to free flights and to reflections. It governs the time evolution of infinitesimally small deviations from the trajectory. Let $(\vec{x},\vec{v})$ and $(\vec{x}\,',\vec{v}\,')$ denote initial and final phase space points of a free flight, respectively. The corresponding tangent map has elements

$$\frac{\partial x_j'}{\partial x_k}=\delta_{jk}, \quad \frac{\partial x_j'}{\partial v_k}=t\,\delta_{jk}, \quad \frac{\partial v_j'}{\partial x_k}=0,$$

$$\frac{\partial v_j'}{\partial v_k}=\delta_{jk}, \quad j,k=1,2,3.$$

To describe a reflection at the cylindrical boundary element of radius r we choose coordinates such that the cylinder axis is parallel to the z axis. Let $(\vec{x},\vec{v})$ and $(\vec{x}\,',\vec{v}\,')$ denote phase space points immediately before and after a reflection, respectively. The corresponding tangent map has elements

$$\frac{\partial x_j'}{\partial x_k}=\delta_{jk}-2n_jn_k, \quad \frac{\partial x_j'}{\partial v_k}=0, \quad j,k=1,2,3,$$

$$\frac{\partial v_j'}{\partial v_k}=\delta_{jk}-2n_jn_k, \quad \frac{\partial v_j'}{\partial x_3}=0, \quad \frac{\partial v_3'}{\partial x_k}=0, \quad j,k=1,2,3,$$

$$\frac{\partial v_j'}{\partial x_k}=-\frac{2}{r}\left(w_n\delta_{jk}+n_jw_k-n_kw_j-\frac{w^2}{w_n}n_jn_k\right), \quad j,k=1,2,$$

where $\vec{n}$ is the outward pointing unit normal vector at the boundary, $\vec{w}=(v_1,v_2)$ is the velocity in the plane normal to the z axis and its normal component $w_n=\vec{n}\cdot\vec{w}$. A reflection at the flat parts of the boundary may be described using the equations above after taking $r\to\infty$.

In a conservative system with three degrees of freedom the Lyapunov exponents come in pairs $(\lambda_j,\lambda_{-j}), j=1,2,3$, with $\lambda_1\geqslant\lambda_2\geqslant\lambda_3=0$ and $\lambda_j+\lambda_{-j}=0$. Our numerical results (mean values, variances, maximal and minimal values) are listed in Table I. We observe two positive Lyapunov exponents, thus indicating that truly high-dimensional chaos has developed. Note that all of the followed trajectories have positive Lyapunov exponents λ_1 and λ_2. We checked our numerical results by comparing forward with backward evolution and by using the alternative method pioneered by Benettin *et al.* [20]. Within our numerical accuracy we found one pair of vanishing Lyapunov exponents and confirmed that the sum of conjugated exponents vanishes. We repeated the computation for a larger ensemble of 10^5 trajectories and a shorter time evolution of about 5×10^4 bounces off the boundary. This computation reproduced the mean values of Table I but the distributions were broader. This is due to the shorter time evolution, which leads to somewhat less converged Lyapunov exponents. Again, no single stable trajectory was found. Therefore, our numerical results strongly suggest that the system under consideration is completely chaotic.

TABLE I. Results for Lyapunov exponents (mean values, variances, minimal and maximal values) obtained from an ensemble of 10^4 runs. All quantities are given in units of $1/r$.

j	λ_j	$\Delta\lambda_j$	$\lambda_{min}^{(j)}$	$\lambda_{max}^{(j)}$
1	0.364	0.001	0.347	0.368
2	0.330	0.001	0.314	0.334

It is interesting to investigate the stability of lower-dimensional invariant manifolds. Such manifolds exist in systems with discrete symmetries and in rotationally invariant many-body systems composed of identical particles [21], though they need not necessarily be connected to a discrete symmetry [15]. Since their stability properties may deviate considerably from the system average, it is important to investigate them more closely. In what follows let us consider two lower-dimensional invariant manifolds, namely, (i) $y=0, p_y=0$ and (ii) $y=\pm r/\sqrt{2}, p_y=0$ or $z=r/\sqrt{2}, p_z=0$. Note that manifold (i) is a symmetry plane of the billiard whereas manifold (ii) is a less trivial example of a low-dimensional invariant manifold. Note further that manifold (i) and manifold (ii) can be viewed as a partly desymmetrized and a fully two-dimensional stadium, respectively. Though these manifolds are of measure zero in phase space, they may exhibit special stability properties in transverse directions [21,4]. This behavior may cause wave function scarring upon quantization [15,22,7]. We start 1000 randomly drawn trajectories inside each of the invariant manifolds and compute the Lyapunov spectrum by following their time evolution for about 5×10^5 collisions with the boundary. One pair of Lyapunov exponents describes the stability in directions transverse to the manifold while the remaining two pairs correspond to the inside motion. The results listed in Table II show that both invariant manifolds are unstable inside and in the transverse direction. A comparison with Table I shows that the local instability close to the invariant manifolds deviates from the average instability inside the

TABLE II. Results for Lyapunov exponents (mean values and variances) for invariant manifolds, obtained from ensembles of 10^3 runs. $\lambda_{||}$ and $\lambda_{\perp}$ denote the Lyapunov exponents inside and transverse to the corresponding invariant manifold. All quantities are given in units of $1/r$.

Manifold	$\lambda_{\|\|}$	$\lambda_{\perp}$
(i)	0.430 ± 0.003	0.305 ± 0.002
(ii)	0.391 ± 0.004	0.362 ± 0.004

billiard. This is not surprising since stability properties of invariant sets like periodic orbits or low-dimensional manifolds fluctuate around the system average. In low-dimensional open systems such a behavior may have considerable influence on quantum transport [23]. Our results hint at a generalization of these observations to three dimensions. We note that the Lyapunov exponent inside each manifold agrees with the one reported for the corresponding two-dimensional stadium billiard [24].

We now turn to the more general case $a>0$ and $r_1 \neq r_2$. To be definite we fix $a=1$, $r_1=\sqrt{2}$, $r_2=\sqrt{3}$ and compute the Lyapunov spectrum from 10^4 trajectories with uniformly distributed random initial conditions and a time evolution of about 5×10^5 bounces off the boundary. As before, we do not find a single stable trajectory and both Lyapunov exponents are positive, i.e., $\lambda_1=0.185\pm 0.001$, $\lambda_2=0.157 \pm 0.001$ in units of $1/a$. This shows that truly high-dimensional chaos exists for these parameter values, too.

Let us also discuss focusing billiards in more than three dimensions. Bunimovich's and Rehacek's construction has successfully been used to create chaos in four-dimensional billiards [4], and it works in higher dimensions as well [5]. It has the advantage that a single spherical cap attached to a billiard with otherwise flat boundaries may be sufficient to render the system chaotic. This is different with the cylindrical elements used in this work. While we do not see any argument in principle that would prohibit the generation of chaos in high-dimensional billiards by means of cylindrical components (i.e., such as are focusing in a two-dimensional plane only), it certainly requires several of such boundary elements to generate the desired degree of local instability. The billiard model of a self-bound interacting many-body system [25] might be a promising candidate for such a scenario. However, more work is necessary for a better understanding of focusing cylindrical boundary elements in high-dimensional systems.

Finally, we want to comment on the role of bouncing ball orbits in the three-dimensional generalized stadium billiard. For $a>0$ there is an infinite number of families of bouncing ball orbits, and the situation is similar to the case of the three-dimensional Sinai billiard. Theoretical [14] and experimental [11] studies of this billiard show that the bouncing ball modes [26] dominate the length spectrum (i.e., the Fourier transform of the fluctuating part of the spectral density). In three dimensions, the amplitude of each bouncing ball mode is enhanced by $O(k)$ (k being the wave vector) when compared with the amplitude of an unstable periodic orbit; an infinite number of bouncing ball modes with different length thus dominates the length spectrum at practically all lengths. This makes the semiclassical analysis of level spectra in terms of periodic orbits a difficult task. The situation is, however, different for $a=0$ and $r_1=r_2=r$. In this case, there are only two families of bouncing ball orbits having equal length $4r$. Thus, the billiard considered in this work is a promising candidate for further experimental and theoretical investigations of wave chaotic phenomena in three dimensions.

In summary, we have studied a generalized three-dimensional stadium billiard that is chaotic due to the defocusing mechanism. The construction uses cylindrical components as focusing boundary elements and thereby differs from the one proposed by Bunimovich and Rehacek. We presented strong numerical evidence that the system considered displays hard chaos. In particular, we found two positive Lyapunov exponents and confirmed the instability of lower-dimensional invariant manifolds.

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